J. Fluid Mech. (1965), *vol.* 22, *part* 3, *pp.* 417–431
Printed in Great Britain

Hypersonic non-equilibrium flow over slender bodies

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(Received 16 October 1964 and in revised form 14 January 1965)

An analytical study is made of non-equilibrium effects on hypersonic, inviscid flow over slender, axisymmetric bodies. Also, two-dimensional results are obtained for the purpose of comparison. The rate process under consideration is that of molecular vibration of the gas. The exact problem is solved by successive approximations based on a double-expansion scheme involving two small parameters: one represents the fact that the bodies considered are slender; the other represents the fact that the vibrational internal energy is small in comparison to the total enthalpy. The exact differential equations and boundary conditions are simplified to the hypersonic-small-disturbance-approximation form. The unknown quantities in this approximate problem are expanded into series of the small parameter, $(\gamma-1)/(\gamma+1)$, which is $\frac{1}{6}$ for a diatomic gas. In this formulation it is found that the classical hypersonic similitude can be extended by slight modifications to cover the added consideration of vibrational non-equilibrium. The modifications introduced are the normalization of all lengths by the characteristic relaxation length of the gas and the addition of a new dimensionless parameter, which is a measure of the excitation level of the vibrational internal energy in the flow field. Explicit, uniformly valid solutions are obtained for the specific problems of flow over a slender cone and of that over a thin wedge. The successive approximations are carried as far as necessary to show the non-equilibrium effect, which differs in order of magnitude for the various flow quantities. One interesting feature of the solutions is the non-monotonic behaviour in the relaxation of the surface pressure of both the cone and the wedge, in contrast to intuitive expectation. The result for a 20° cone in a free stream of oxygen at 300° K and a Mach number of 15 is displayed and compared with the numerical solution of the exact problem using the method of characteristics.

1. Introduction

Ever since non-equilibrium effects were recognized to be important in high-temperature gas dynamics, considerable effort has been devoted to further study of various classical problems taking account of these new effects. Much understanding has been gained through successful analytical studies of such non-equilibrium flows as those across a normal shock wave, over a wavy wall, over a wedge, etc. Extension of these studies to the more complicated problem of flow over axisymmetric bodies is certainly necessary and natural. Clarke (1960) and Li & Wang (1962), considering only slender bodies, used a linearized approach to analyse the problem. However, in most of the realistic cases the non-equilibrium effects in flow over slender bodies can only be induced by the presence of a strong shock wave, a situation which exists at high free-stream Mach number and which is not covered by the linearized analysis. Therefore, in the present analysis the non-linear problem of hypersonic non-equilibrium flow over slender bodies is considered.

On the other hand, numerical analyses using the method of characteristics have been made on some supersonic non-equilibrium flows. Of particular interest to the present study is the work on flow over a cone by Sedney & Gerber (1963). They found that the surface pressure, instead of relaxing monotonically toward its equilibrium value far downstream from the leading edge, undergoes an over-expansion during its relaxation. It is implied in their work that this seemingly anomalous behaviour, not to be expected intuitively, is due to the axisymmetric nature of the flow over a cone. To clear up this point and to answer the general question of what are the differences, if any, between non-equilibrium effects on the flow over a two-dimensional and an axisymmetric body, an analytical study is definitely required.

The present problem is formulated for a gas capable of relaxing in one vibrational mode of internal energy. Vibrational non-equilibrium has a simpler rate process than, say, that of dissociation and hence lends itself more readily to mathematical treatment. Also, it is the same non-equilibrium process considered by Sedney & Gerber. Naturally, the same process has to be considered here in order to compare the present results with those obtained by numerical analysis. A systematic double-expansion scheme, developed by Cole (1957) for the similar flow of a perfect gas, is employed to treat the problem. This scheme is built on two small parameters: one represents the fact that the bodies considered are slender; the other represents, in the context of the present work, the fact that the vibrational internal energy is small compared with the total enthalpy. The exact differential equations and boundary conditions are thus, as a first step, simplified to the first approximation of the hypersonic small-disturbance theory. The higher approximations of this expansion need not concern us for the same reason as in the case of a perfect gas. The classical hypersonic similitude is found to be only slightly modified by the added consideration of vibrational non-equilibrium. The modifications are the normalization of all lengths by the characteristic relaxation length of the gas and the addition of a new dimensionless parameter, which is a measure of the excitation level of the vibrational internal energy in the flow field. To solve the approximated problem, a second expansion is taken in terms of the small parameter $(\gamma-1)/(\gamma+1)$, which is $\frac{1}{6}$ for a diatomic gas. This step uncouples the rate equation from the conservation equations and amounts to a perturbation about the flow in which the vibrational internal energy is frozen.

Although bodies of general shape are considered in the formulation of the problem, solutions are carried out only for the specific cases of a slender cone and a thin wedge. It is found that for various flow quantities the non-equilibrium effects differ in order of magnitude. The successive approximations are carried as far as necessary to show the non-equilibrium effects. The solutions are explicit and uniformly valid throughout the flow field. The non-equilibrium effect enters the solutions as a relaxation term which changes the value of each flow quantity from its frozen-flow value just behind the shock wave to its equilibrium-flow value far downstream. Interestingly enough, the surface pressure on both the cone and the wedge is found to over-expand during its relaxation. This finding certainly contradicts the implication of Sedney & Gerber. The result for a 20° cone in a free stream of oxygen at 300 °K and a Mach number of 15 is displayed and shown in good agreement with the numerical solution of the exact problem carried out by Sedney & Gerber using the method of characteristics.

2. Exact problem

The equations expressing conservation of mass, momentum and energy for steady, two-dimensional or axisymmetric flow of an inviscid, non-heat-conducting gas are

$$\frac{\partial}{\partial x}(r^j\rho u)+\frac{\partial}{\partial r}(r^j\rho v) = 0, \quad j = \begin{Bmatrix} 0 & \text{two-dimensional} \\ 1 & \text{axisymmetric} \end{Bmatrix}, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{3}$$

$$u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial r}-\frac{1}{\rho}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial r}\right) = 0, \tag{4}$$

where x is measured along and r perpendicular to the body axis, and u and v are the velocities in the x- and r-directions, respectively (figure 1); p is the pressure, ρ the density and h the enthalpy. If the gas is capable of being excited in its vibrational mode of internal energy, the equations of state are

$$p = R\rho T, \tag{5}$$

$$h = \frac{\gamma}{(\gamma-1)}\frac{p}{\rho}+e_v, \tag{6}$$

where T is the temperature, R the gas constant, γ the ratio of frozen specific heats and e_v denotes the vibrational energy. When the vibrational energy is not in equilibrium, a rate equation is needed and takes the form

$$u\frac{\partial e_v}{\partial x}+v\frac{\partial e_v}{\partial r} = \frac{e_v^*-e_v}{\tau}. \tag{7}$$

In (7), e_v^* is the fictitious value that e_v would assume if the gas were in equilibrium at the local temperature, and is given as a function of temperature by

$$e_v^* = \frac{R\theta_v}{e^{\theta_v/T}-1}, \tag{8}$$

where the constant θ_v is the characteristic temperature of molecular vibration. The relaxation time τ is a very complicated function of pressure and temperature; however, in the present problem its value varies only slightly. The analysis as well as the results may be simplified without losing the essential physical features of the problem if τ is treated as a constant. The number of equations involved can be reduced by using the equations of state to eliminate two thermodynamic variables from the conservation and rate equations. If one chooses to work with

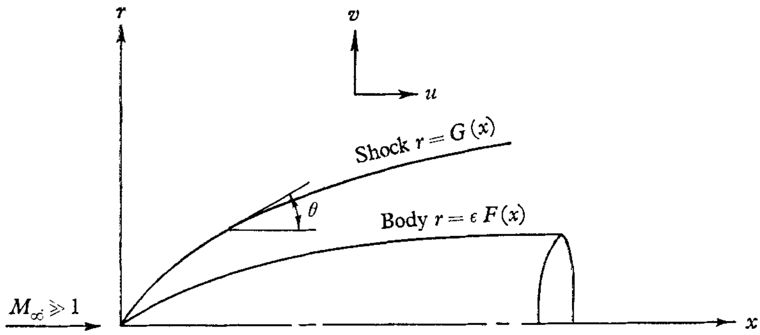

FIGURE 1. The geometry of the flow.

p, ρ and e_v, as the thermodynamic variables, then (4) and (7) may be rewritten, by virtue of (5), (6) and (8), as

$$u\frac{\partial}{\partial x}\left(\frac{p}{\rho^{\gamma}}\right)+v\frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right)+\frac{(\gamma-1)}{\rho^{\gamma-1}}\left(u\frac{\partial e_v}{\partial x}+v\frac{\partial e_v}{\partial r}\right)=0, \tag{9}$$

$$\tau\left(u\frac{\partial e_v}{\partial x}+v\frac{\partial e_v}{\partial r}\right)=\frac{R\theta_v}{e^{R\theta_v\rho/p}-1}-e_v. \tag{10}$$

If the unknown shock-wave shape is described by $r = G(x)$ and if the free stream is assumed to be in equilibrium, the frozen situation behind the shock wave is given by the following shock conditions:

$$u[x,G(x)]=u_\infty\left[1-2\frac{M_\infty^2\sin^2\theta-1}{(\gamma+1)M_\infty^2}\right], \tag{11}$$

$$v[x,G(x)]=u_\infty\left[2\frac{M_\infty^2\sin^2\theta-1}{(\gamma+1)M_\infty^2}\right]\cot\theta, \tag{12}$$

$$p[x,G(x)]=p_\infty\left[\frac{2\gamma M_\infty^2\sin^2\theta-(\gamma-1)}{(\gamma+1)}\right], \tag{13}$$

$$\rho[x,G(x)]=\rho_\infty\left[\frac{(\gamma+1)M_\infty^2\sin^2\theta}{2+(\gamma-1)M_\infty^2\sin^2\theta}\right], \tag{14}$$

$$e_v[x,G(x)]=\frac{R\theta_v}{\exp(R\theta_v\rho_\infty/p_\infty)-1}, \tag{15}$$

where the subscript ∞ refers to free-stream conditions, M is the frozen Mach number and θ is the shock angle (figure 1) related to the shock-wave shape through

$$\theta=\tan^{-1}G'(x), \tag{16}$$

in which the prime denotes differentiation with respect to the argument. Now consider a given slender body described by

$$r=\epsilon F(x), \tag{17}$$

where ϵ is the slenderness parameter, assumed small ($\epsilon \ll 1$). The tangency condition at the body surface requires that

$$\frac{v[x, \epsilon F(x)]}{u[x, \epsilon F(x)]} = \epsilon \frac{dF(x)}{dx} \equiv \epsilon F'(x). \tag{18}$$

To define completely the shock-wave shape, the condition

$$G(0) = 0, \tag{19}$$

expressing the requirement that the shock wave is attached at the leading edge, is imposed (figure 1). The five equations (1), (2), (3), (9) and (10) and the boundary conditions (11) through (19) are to be solved for the unknowns, u, v, p, ρ, e_v and $G(x)$, which give the complete flow field.

3. Hypersonic small-disturbance approximation and similitude

In posing the exact problem, interest has been limited to consideration of slender bodies. Further attention will be limited to hypersonic flows, for which $M_\infty \gg 1$, while

$$M_\infty \epsilon \equiv K = O(1). \tag{20}$$

The exact problem can then be simplified by using the hypersonic small-disturbance approximation. The variables will be expanded in series of ϵ, as functions of the distorted co-ordinates.

$$\bar{\xi} = x/(u_\infty \tau); \quad \bar{\eta} = r/(u_\infty \tau \epsilon), \tag{21}$$

in which τ is considered to be a constant evaluated at the leading edge behind the shock wave and $u_\infty \tau$ represents the relaxation length of the gas. The expansions are the following

$$u(x,r) = u_\infty[1+\epsilon^2 \bar{u}(\bar{\xi},\bar{\eta}) + O(\epsilon^4)], \tag{22}$$

$$v(x,r) = u_\infty \epsilon[\bar{v}(\bar{\xi},\bar{\eta}) + O(\epsilon^2)], \tag{23}$$

$$p(x,r) = \rho_\infty u_\infty^2 \epsilon^2[\bar{p}(\bar{\xi},\bar{\eta}) + O(\epsilon^2)], \tag{24}$$

$$\rho(x,r) = \rho_\infty[\bar{\rho}(\bar{\xi},\bar{\eta}) + O(\epsilon^2)], \tag{25}$$

$$e_v(x,r) = u_\infty^2 \epsilon^2[\bar{e}_v(\bar{\xi},\bar{\eta}) + O(\epsilon^2)], \tag{26}$$

$$G(x) = u_\infty \tau \epsilon[\bar{g}(\bar{\xi}) + O(\epsilon^2)]. \tag{27}$$

Since the error contained in the barred quantities is of the order of ϵ^2, there is practically no need to investigate the higher approximations. Moreover, it will be seen later that the barred quantities do contain all of the necessary physical features of the flow field.

Substituting equations (22) through (27) into the exact problem and neglecting the higher-order terms leads to the following approximate problem:

$$\frac{\partial}{\partial \bar{\xi}}(\bar{\eta}^j \bar{\rho}) + \frac{\partial}{\partial \bar{\eta}}(\bar{\eta}^j \bar{\rho} \bar{v}) = 0, \tag{28}$$

$$\frac{\partial \bar{u}}{\partial \bar{\xi}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{\eta}} + \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{\xi}} = 0, \tag{29}$$

$$\frac{\partial \bar{v}}{\partial \bar{\xi}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{\eta}} + \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{\eta}} = 0, \tag{30}$$

$$\left(\frac{\partial}{\partial \bar{\xi}}+\bar{v}\frac{\partial}{\partial \bar{\eta}}\right)\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right)+\frac{(\gamma-1)}{\bar{\rho}^{\gamma-1}}\left(\frac{\partial \bar{e}_v}{\partial \bar{\xi}}+\bar{v}\frac{\partial \bar{e}_v}{\partial \bar{\eta}}\right) = 0, \tag{31}$$

$$\frac{\partial \bar{e}_v}{\partial \bar{\xi}}+\bar{v}\frac{\partial \bar{e}_v}{\partial \bar{\eta}} = \frac{\Theta}{\exp(\Theta\bar{\rho}/\bar{p})-1}-\bar{e}_v, \tag{32}$$

where
$$\Theta \equiv \frac{R\theta_v}{u_\infty^2\,\epsilon^2} = \left(\frac{\theta_v}{T_\infty}\right)\left(\frac{1}{\gamma K^2}\right),$$

and
$$\bar{u}[\bar{\xi},\bar{g}(\bar{\xi})] = [2/(\gamma+1)K^2](1-K^2\bar{g}'^2), \tag{33}$$

$$\bar{v}[\bar{\xi},\bar{g}(\bar{\xi})] = [2/(\gamma+1)K^2\bar{g}'](K^2\bar{g}'^2-1), \tag{34}$$

$$\bar{p}[\bar{\xi},\bar{g}(\bar{\xi})] = [2\gamma K^2\bar{g}'^2-(\gamma-1)]/[\gamma(\gamma+1)K^2], \tag{35}$$

$$\bar{\rho}[\bar{\xi},\bar{g}(\bar{\xi})] = [(\gamma+1)K^2\bar{g}'^2]/[2+(\gamma-1)K^2\bar{g}'^2], \tag{36}$$

$$\bar{e}_v[\bar{\xi},\bar{g}(\bar{\xi})] = \Theta/(e^{\gamma\Theta K^2}-1), \tag{37}$$

$$\bar{v}[\bar{\xi},\bar{f}(\bar{\xi})] = \bar{f}'(\bar{\xi}), \tag{38}$$

$$\bar{g}(0) = 0, \tag{39}$$

where
$$\bar{f}(\bar{\xi}) = [F(x)]/(u_\infty\tau) = [F(u_\infty\tau\bar{\xi})]/(u_\infty\tau).$$

The simplifications thus achieved are, besides those in the boundary conditions, the uncoupling of the x-momentum equation from the rest and the replacing of the unknown streamwise velocity by the constant free-stream velocity. Yet the essential non-linearity of the problem is still preserved.

A simple similitude accompanies this approximation. The structure of equations (28) through (39) enables one to write in functional form,† for instance,

$$C_p = \frac{p}{\rho_\infty u_\infty^2} = \epsilon^2\pi\left(\frac{x}{u_\infty\tau},\frac{r}{u_\infty\tau\epsilon};\gamma,\frac{u_\infty^2\,\epsilon^2}{RT_\infty},\frac{\theta_v}{T_\infty}\right), \tag{40}$$

which is only a slight modification of the hypersonic similitude for a perfect gas (see, for example, Van Dyke 1954). Now the co-ordinates are non-dimensionalized by the relaxation length $u_\infty\tau$; also the new parameter θ_v/T_∞ is added which can be viewed as a measure of the excitation level of the vibrational energy in the flow field. Naturally, with the additional consideration of vibrational non-equilibrium, this similitude is more restrictive than the classical one; nevertheless, it is still useful because according to it, simulation of, say, free-stream conditions is possible.

4. Expansion with respect to $(\gamma-1)/(\gamma+1)$

The system (28) through (32) is too complicated to be integrated analytically, especially due to the coupling between the energy and the rate equation. One technique to overcome this difficulty in solving non-equilibrium problems is to apply a perturbation about the frozen-flow solution. In this instance the perturbation can be accomplished by taking advantage of the fact that the vibrational energy is a small fraction of the total enthalpy (see Lee 1964). Mathematically, the perturbation involves the parameter γ, which is close to unity.

† This form is restricted to bodies without a characteristic length. For bodies with characteristic length L, another parameter $L/u_\infty\tau$ will appear in the expression $\bar{f}(\bar{\xi})$ and should be included in (40).

In (31) the terms contributed by the vibrational energy are proportional to $(\gamma-1)$, and are hence small and negligible in a first approximation.

In the subsequent expansion scheme the first approximation is known classically as the Newtonian solution (see Cole 1957), which has an infinitesimally thin shock layer. In order to perturb the Newtonian solution, a magnified scale has to be used such that the details of the thin layer can prevail. As is necessary for the magnification, a transformation should be made which measures the transverse distance from the body surface and the transverse velocity from its value on the body surface. Let

$$\bar{\xi} = \tilde{\xi}, \quad \bar{\eta} - \bar{f}(\bar{\xi}) = \tilde{\eta}, \tag{41}$$

$$\bar{u}(\bar{\xi},\bar{\eta}) = \tilde{u}(\tilde{\xi},\tilde{\eta}), \tag{42}$$

$$\bar{v}(\bar{\xi},\bar{\eta}) - \bar{f}'(\bar{\xi}) = \tilde{v}(\tilde{\xi},\tilde{\eta}), \tag{43}$$

$$\bar{p}(\bar{\xi},\bar{\eta}) = \tilde{p}(\tilde{\xi},\tilde{\eta}), \tag{44}$$

$$\bar{\rho}(\bar{\xi},\bar{\eta}) = \tilde{\rho}(\tilde{\xi},\tilde{\eta}), \tag{45}$$

$$\bar{e}_v(\bar{\xi},\bar{\eta}) = \tilde{e}_v(\tilde{\xi},\tilde{\eta}), \tag{46}$$

$$\bar{g}(\bar{\xi}) - \bar{f}(\bar{\xi}) = \tilde{g}(\tilde{\xi}). \tag{47}$$

The problem given by (28) through (39) becomes

$$\frac{1}{\tilde{\rho}}\left(\frac{\partial\tilde{\rho}}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial\tilde{\rho}}{\partial\tilde{\eta}}\right) + \frac{\partial\tilde{v}}{\partial\tilde{\eta}} + j\left[\frac{\tilde{v}+\bar{f}'(\tilde{\xi})}{\tilde{\eta}+\bar{f}(\tilde{\xi})}\right] = 0, \tag{48}$$

$$\frac{\partial\tilde{u}}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{\eta}} + \frac{1}{\tilde{\rho}}\left[\frac{\partial\tilde{p}}{\partial\tilde{\xi}} - \bar{f}'(\tilde{\xi})\frac{\partial\tilde{p}}{\partial\tilde{\eta}}\right] = 0, \tag{49}$$

$$\frac{\partial\tilde{v}}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{\eta}} + \bar{f}''(\tilde{\xi}) + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{p}}{\partial\tilde{\eta}} = 0, \tag{50}$$

$$\left(\frac{\partial}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial}{\partial\tilde{\eta}}\right)\left(\frac{\tilde{p}}{\tilde{\rho}^{\gamma}}\right) + \frac{(\gamma-1)}{\tilde{\rho}^{\gamma-1}}\left(\frac{\partial}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial}{\partial\tilde{\eta}}\right)\tilde{e}_v = 0, \tag{51}$$

$$\frac{\partial\tilde{e}_v}{\partial\tilde{\xi}} + \tilde{v}\frac{\partial\tilde{e}_v}{\partial\tilde{\eta}} = \frac{\Theta}{\exp(\Theta\tilde{\rho}/\tilde{p}) - 1} - \tilde{e}_v \tag{52}$$

and

$$\tilde{u}[\tilde{\xi},\tilde{g}(\tilde{\xi})] = [2/(\gamma+1)K^2][1 - K^2(\bar{f}'+\tilde{g}')^2], \tag{53}$$

$$\tilde{v}[\tilde{\xi},\tilde{g}(\tilde{\xi})] = -\bar{f}'(\tilde{\xi}) + [2/(\gamma+1)K^2(\bar{f}'+\tilde{g}')][K^2(\bar{f}'+\tilde{g}')^2 - 1], \tag{54}$$

$$\tilde{p}[\tilde{\xi},\tilde{g}(\tilde{\xi})] = [2\gamma K^2(\bar{f}'+\tilde{g}')^2 - (\gamma-1)]/[\gamma(\gamma+1)K^2], \tag{55}$$

$$\tilde{\rho}[\tilde{\xi},\tilde{g}(\tilde{\xi})] = [(\gamma+1)K^2(\bar{f}'+\tilde{g}')^2]/[2+(\gamma-1)K^2(\bar{f}'+\tilde{g}')^2], \tag{56}$$

$$\tilde{e}_v[\tilde{\xi},\tilde{g}(\tilde{\xi})] = \Theta/[e^{\gamma\Theta K^2} - 1], \tag{57}$$

$$\tilde{v}[\tilde{\xi},0] = 0, \tag{58}$$

$$\tilde{g}(0) = -\bar{f}(0) = 0. \tag{59}$$

Now consider the small parameter $\delta \equiv (\gamma-1)/(\gamma+1)$ which is equal to $\frac{1}{6}$ for a diatomic gas. In terms of it the tilde quantities can be expanded. The expansions are made in the again distorted co-ordinates

$$\xi = \tilde{\xi}; \quad \eta = \tilde{\eta}/\delta, \tag{60}$$

as

$$\tilde{u}(\tilde{\xi},\tilde{\eta};\,\gamma,K,\Theta) = u_1(\xi,\eta;\,N,\lambda)+\delta u_2(\xi,\eta;\,N,\lambda)+\delta^2 u_3(\xi,\eta;\,N,\lambda)+\ldots, \tag{61}$$

$$\tilde{v}(\tilde{\xi},\tilde{\eta};\,\gamma,K,\Theta) = \delta v_1(\xi,\eta;\,N,\lambda)+\delta^2 v_2(\xi,\eta;\,N,\lambda)+\delta^3 v_3(\xi,\eta;\,N,\lambda)+\ldots, \tag{62}$$

$$\tilde{p}(\tilde{\xi},\tilde{\eta};\,\gamma,K,\Theta) = p_1(\xi,\eta;\,N,\lambda)+\delta p_2(\xi,\eta;\,N,\lambda)+\delta^2 p_3(\xi,\eta;\,N,\lambda)+\ldots, \tag{63}$$

$$\tilde{\rho}(\tilde{\xi},\tilde{\eta};\,\gamma,K,\Theta) = (1/\delta)\rho_1(\xi,\eta;\,N,\lambda)+\rho_2(\xi,\eta;\,N,\lambda)+\delta\rho_3(\xi,\eta;\,N,\lambda)+\ldots, \tag{64}$$

$$\tilde{e}_v(\tilde{\xi},\tilde{\eta};\,\gamma,K,\Theta) = \delta e_{v_1}(\xi,\eta;\,N,\lambda)+\delta^2 e_{v_2}(\xi,\eta;\,N,\lambda)+\delta^3 e_{v_3}(\xi,\eta;\,N,\lambda)+\ldots, \tag{65}$$

$$\tilde{g}(\tilde{\xi};\,\gamma,K,\Theta) = \delta g_1(\xi;\,N,\lambda)+\delta^2 g_2(\xi;\,N,\lambda)+\delta^3 g_3(\xi;\,N,\lambda)+\ldots, \tag{66}$$

where N and λ are two parameters considered to be of order unity and defined as

$$N \equiv \frac{1}{K^2\delta}\left(=\frac{(\gamma+1)}{(\gamma-1)\,M_\infty^2\,\epsilon^2}\right) = O(1), \tag{67}$$

$$\lambda \equiv \frac{\Theta}{\delta}\left(=\frac{(\gamma+1)\,\theta_v}{\gamma(\gamma-1)\,T_\infty\,M_\infty^2\,\epsilon^2}\right) = O(1). \tag{68}$$

Substituting the above into (48) through (59) and collecting terms of the same order in δ leads to the following successive approximations.

First approximation

$$\frac{\partial\rho_1}{\partial\xi}+\frac{\partial}{\partial\eta}(\rho_1 v_1)+j\rho_1\frac{\bar{f}'(\xi)}{\bar{f}(\xi)} = 0, \tag{69}$$

$$\frac{\partial u_1}{\partial\xi}+v_1\frac{\partial u_1}{\partial\eta}-\frac{\bar{f}'(\xi)}{\rho_1}\frac{\partial p_1}{\partial\eta} = 0, \tag{70}$$

$$\bar{f}''(\xi)+\frac{1}{\rho_1}\frac{\partial p_1}{\partial\eta} = 0, \tag{71}$$

$$\left(\frac{\partial}{\partial\xi}+v_1\frac{\partial}{\partial\eta}\right)\left(\frac{p_1}{\rho_1}\right) = 0, \tag{72}$$

$$\frac{\partial e_{v_1}}{\partial\xi}+v_1\frac{\partial e_{v_1}}{\partial\eta} = \frac{\lambda}{e^{\lambda\rho_1/p_1}-1} \tag{73}$$

and

$$u_1[\xi,g_1(\xi)] = -[\bar{f}'(\xi)]^2, \tag{74}$$

$$v_1[\xi,g_1(\xi)] = g_1'-(N/\bar{f}')-\bar{f}', \tag{75}$$

$$p_1[\xi,g_1(\xi)] = [\bar{f}'(\xi)]^2, \tag{76}$$

$$\rho_1[\xi,g_1(\xi)] = [\bar{f}'(\xi)]^2/\{N+[\bar{f}'(\xi)]^2\}, \tag{77}$$

$$e_{v_1}[\xi,g_1(\xi)] = \lambda/[e^{\lambda/N}-1], \tag{78}$$

$$v_1[\xi,0] = 0, \tag{79}$$

$$g_1(0) = 0. \tag{80}$$

In this approximation, the conservation equations (69) through (72) are uncoupled from the rate equation (73) and are exactly the same as those of the first approximation in the work of Cole (1957) for a perfect gas. In other words, the vibrational non-equilibrium does not affect the flow field to a first approximation. Rather, the classical solution of the flow quantities is used to calculate the new variable e_{v_1} by use of the rate equation.

Second approximation

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)\left(\frac{\rho_2}{\rho_1}\right)+\frac{\partial v_2}{\partial \eta}+v_2\frac{\partial}{\partial \eta}(\ln \rho_1)+j\frac{\bar{f}'}{\bar{f}}\left(\frac{v_1}{\bar{f}'}-\frac{\eta}{\bar{f}}\right)=0, \tag{81}$$

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)u_2+v_2\frac{\partial u_1}{\partial \eta}-\frac{1}{\rho_1}\left(\bar{f}'\frac{\partial p_2}{\partial \eta}-\frac{\partial p_1}{\partial \xi}-\frac{\rho_2}{\rho_1}\bar{f}'\frac{\partial p_1}{\partial \eta}\right)=0, \tag{82}$$

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)v_1+\frac{1}{\rho_1}\frac{\partial p_2}{\partial \eta}-\frac{\rho_2}{\rho_1^2}\frac{\partial p_1}{\partial \eta}=0, \tag{83}$$

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)\left(\frac{p_2}{\rho_1}-\frac{p_1\rho_2}{\rho_1^2}-2\frac{p_1}{\rho_1}\ln \rho_1\right)+v_2\frac{\partial}{\partial \eta}\left(\frac{p_1}{\rho_1}\right)+2\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)e_{v_1}=0, \tag{84}$$

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)e_{v_2}+v_2\frac{\partial e_{v_1}}{\partial \eta}=\frac{\lambda^2\rho_1 e^{\lambda\rho_1/p_1}}{p_1(e^{\lambda\rho_1/p_1}-1)^2}\left(\frac{p_2}{p_1}-\frac{\rho_2}{\rho_1}\right)-e_{v_2} \tag{85}$$

and

$$u_2[\xi,g_1(\xi)]+g_2(\xi)\frac{\partial u_1}{\partial \eta}[\xi,g_1(\xi)]=[\bar{f}'(\xi)]^2+N-2\bar{f}'g_1', \tag{86}$$

$$v_2[\xi,g_1(\xi)]+g_2(\xi)\frac{\partial v_1}{\partial \eta}[\xi,g_1(\xi)]=g_2'+\frac{Ng_1'}{[\bar{f}'(\xi)]^2}-g_1'+\frac{N}{\bar{f}'}, \tag{87}$$

$$p_2[\xi,g_1(\xi)]+g_2(\xi)\frac{\partial p_1}{\partial \eta}[\xi,g_1(\xi)]=[\bar{f}'(\xi)]^2\left(2\frac{g_1'}{\bar{f}'}-1\right), \tag{88}$$

$$\rho_2[\xi,g_1(\xi)]+g_2(\xi)\frac{\partial \rho_1}{\partial \eta}[\xi,g_1(\xi)]=\frac{N\bar{f}'(2g_1'+\bar{f}')}{\{N+[\bar{f}'(\xi)]^2\}^2}, \tag{89}$$

$$e_{v_2}[\xi,g_1(\xi)]+g_2(\xi)\frac{\partial e_{v_1}}{\partial \eta}[\xi,g_1(\xi)]=-2\frac{\lambda^2 e^{\lambda/N}}{N(e^{\lambda/N}-1)^2}, \tag{90}$$

$$v_2[\xi,0]=0, \tag{91}$$

$$g_2(0)=0. \tag{92}$$

Third approximation

As it is intended to carry the successive approximations far enough to show the non-equilibrium effect on all flow variables, it will be seen later that some of the equations in this approximation are necessary.

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)u_3+v_2\frac{\partial u_2}{\partial \eta}+v_3\frac{\partial u_1}{\partial \eta}+\frac{1}{\rho_1}\frac{\partial p_2}{\partial \xi}-\frac{\rho_2}{\rho_1^2}\frac{\partial p_1}{\partial \xi}$$
$$-\frac{\bar{f}'}{\rho_1}\left[\frac{\partial p_3}{\partial \eta}-\frac{\rho_2}{\rho_1}\frac{\partial p_2}{\partial \eta}+\left(\frac{\rho_2^2}{\rho_1^2}-\frac{\rho_3}{\rho_1}\right)\frac{\partial p_1}{\partial \eta}\right]=0, \tag{93}$$

$$\left(\frac{\partial}{\partial \xi}+v_1\frac{\partial}{\partial \eta}\right)v_2+v_2\frac{\partial v_1}{\partial \eta}+\frac{1}{\rho_1}\frac{\partial p_3}{\partial \eta}-\frac{\rho_2}{\rho_1^2}\frac{\partial p_2}{\partial \eta}+\left(\frac{\rho_2^2}{\rho_1^3}-\frac{\rho_3}{\rho_1^2}\right)\frac{\partial p_1}{\partial \eta}=0, \tag{94}$$

and
$$u_3[\xi, g_1(\xi)] + g_2(\xi)\frac{\partial u_2}{\partial \eta}[\xi, g_1(\xi)] + g_3(\xi)\frac{\partial u_1}{\partial \eta}[\xi, g_1(\xi)] + \frac{[g_2(\xi)]^2}{2}\frac{\partial^2 u_1}{\partial \eta^2}[\xi, g_1(\xi)] = 2\bar{f}'g_1' - g_1'^2 - 2\bar{f}'g_2' - N, \tag{95}$$

$$p_3[\xi, g_1(\xi)] + g_2(\xi)\frac{\partial p_2}{\partial \eta}[\xi, g_1(\xi)] + g_3(\xi)\frac{\partial p_1}{\partial \eta}[\xi, g_1(\xi)] + \frac{[g_2(\xi)]^2}{2}\frac{\partial^2 p_1}{\partial \eta^2}[\xi, g_1(\xi)] = 2\bar{f}'g_2' + g_1'^2 - 2\bar{f}'g_1' - N. \tag{96}$$

5. Solutions for wedge and cone

Although the problem is formulated for slender bodies of general shape, the solutions for a wedge and a cone will now be carried out, for which

$$F(x) = x; \quad \bar{f}(\xi) = \xi; \quad \bar{f}'(\xi) = 1; \quad \bar{f}''(\xi) = 0$$

and $\epsilon = \tan\beta$ where β denotes the semi-vertex angle. The solution of the approximate problems is simple and straightforward because the equations of each approximation can be integrated one by one. The details of the integration will not be given here.

Solving the first approximation for wedge and cone yields

$$p_1 = 1, \quad u_1 = -1, \quad \rho_1 = (N+1)^{-1}, \qquad (97), (98), (99)$$

$$v_1 = 0 \quad \text{wedge} \qquad (100a)$$

$$= -(\eta/\xi) \quad \text{cone}, \qquad (100b)$$

$$g_1 = (N+1)\,\xi \quad \text{wedge} \qquad (101a)$$

$$= \tfrac{1}{2}(N+1)\,\xi \quad \text{cone}, \qquad (101b)$$

$$e_{v_1} = \frac{\lambda}{e^{\lambda/(N+1)}-1}\left\{1+\left[\frac{e^{\lambda/(N+1)}-1}{e^{\lambda/N}-1}-1\right]\exp\left\{-\xi\left[1-\frac{\eta}{(N+1)\,\xi}\right]\right\}\right\} \quad \text{wedge} \qquad (102a)$$

$$= \frac{\lambda}{e^{\lambda/(N+1)}-1}\left\{1+\left[\frac{e^{\lambda/(N+1)}-1}{e^{\lambda/N}-1}-1\right]\exp\left\{-\xi\left[1-\left(\frac{2\eta}{(N+1)\,\xi}\right)^{\frac{1}{2}}\right]\right\}\right\}, \quad \text{cone}. \qquad (102b)$$

The results show that to this approximation the pressure, density, velocities an shock-wave shape are not affected by the non-equilibrium effect yet and that they are the Newtonian solutions as known classically. The streamlines in the wedge case are straight lines parallel to the wedge surface and are described by η = const. The streamlines in the cone case are hyperbolas described by $\xi\eta$ = const. The non-equilibrium effect is shown on the vibrational energy which, starting from the frozen (free-stream) value at the shock, relaxes (decays exponentially) along the streamlines to the equilibrium value far downstream.

In order to observe the non-equilibrium effect on the other variables, it is necessary to go to the higher approximations. Solving the second approximation yields

$$p_2 = 2N+1 \quad \text{wedge} \quad (103a)$$

$$= \tfrac{1}{4}(5N+1)-(\eta/\xi)^2/(N+1) \quad \text{cone,} \quad (103b)$$

$$u_2 = -(N+1) \quad \text{wedge} \quad (104a)$$

$$= -[\tfrac{1}{2}(N+1)-(\eta/\xi)] \quad \text{cone,} \quad (104b)$$

$$\rho_2 = \frac{2}{(N+1)^2}\left\{\frac{N(2N+3)}{2}+\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left[1-\exp\left[-\xi\left(1-\frac{\eta}{(N+1)\,\xi}\right)\right]\right]\right\} \quad \text{wedge} \quad (105a)$$

$$= \frac{1}{(N+1)^2}\left\{\frac{5N^2+10N+1}{4}-\left(\frac{\eta}{\xi}\right)^2 +2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left[1-\exp\left[-\xi\left(1-\left\{\frac{2\eta}{(N+1)\,\xi}\right\}^{\frac{1}{2}}\right)\right]\right]\right\} \quad \text{cone,} \quad (105b)$$

$$v_2 = -2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left\{[1-e^{-\xi}]-\left[1-\exp\left[-\xi\left(1-\frac{\eta}{(N+1)\,\xi}\right)\right]\right]\right\} \quad \text{wedge} \quad (106a)$$

$$= \left(\frac{\eta}{\xi}\right)^2-\frac{4}{3(N+1)}\left(\frac{\eta}{\xi}\right)^3-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left\{\left[1-\frac{(1-e^{-\xi})}{\xi}\right] -\left[1-\left\{\frac{2\eta}{(N+1)\,\xi}\right\}^{\frac{1}{2}}\exp\left[-\xi\left(1-\left\{\frac{2\eta}{(N+1)\,\xi}\right\}^{\frac{1}{2}}\right)\right]\right] +\frac{1}{\xi}-\frac{1}{\xi}\exp\left[-\xi\left(1-\left\{\frac{2\eta}{(N+1)\,\xi}\right\}^{\frac{1}{2}}\right)\right]\right\} \quad \text{cone,} \quad (106b)$$

$$g_2 = -(N^2+N-1)\,\xi-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)(\xi+e^{-\xi}-1) \quad \text{wedge} \quad (107a)$$

$$= \frac{1}{2}\left[1-\frac{5(N+1)^2}{12}\right]\xi+2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left(1-\frac{\xi}{2}+\frac{e^{-\xi}-1}{\xi}\right) \quad \text{cone.} \quad (107b)$$

The expressions (105), (106) and (107) for the density, the lateral velocity and the shock-wave shape, respectively, consist of two parts. One is associated with the non-equilibrium effect and is characterized by the factor

$$\lambda([e^{\lambda/(N+1)}-1]^{-1}-[e^{\lambda/N}-1]^{-1}),$$

which is the difference between the frozen-flow and the equilibrium-flow value of the vibrational energy. The other part represents the classical correction to the first approximation. For density, the non-equilibrium term increases its value monotonically from zero at the frozen shock-wave to a finite value far downstream, and thus the density relaxes from its frozen-flow value to its equilibrium-flow value. For the lateral velocity, the non-equilibrium term consists of two factors: one represents the relaxation of velocity from the frozen shock-wave like that of density; the other represents the correction due to the fact that the true shock-wave, from which the relaxation should start, departs from the frozen shock-wave. Differentiating (107) yields

$$g_2' = -(N^2+N-1)-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)(1-e^{-\xi}) \quad \text{wedge} \quad (108a)$$

$$= \frac{1}{2}-\frac{5(N+1)^2}{24}-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left(\frac{1}{2}+\frac{e^{-\xi}}{\xi}+\frac{e^{-\xi}-1}{\xi^2}\right) \quad \text{cone,} \quad (108b)$$

from which one can see that the non-equilibrium effect changes monotonically the shock-wave slope from the larger, frozen-flow value at the leading edge to the smaller, equilibrium-flow value far away from the leading edge. Evidently, there is no non-equilibrium term in the expressions for the pressure and the axial velocity which are the least sensitive to the non-equilibrium effect. For the vibrational energy, the non-equilibrium effect has already been in evidence in the first approximation, so the lengthy second approximation, which gives only quantitative correction, will not be given here.

Since the relaxation of pressure is of some interest, especially because of the previously raised question as to whether and when it is monotonic, one must proceed to the third approximation. Solving the equation (94) and using the condition (96) yields

$$p_3 = -(N^2+3N-1)-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left\{(1-2e^{-\xi}) + \xi\left[1-\frac{\eta}{(N+1)\,\xi}\right]e^{-\xi}+\exp\left[-\xi\left(1-\frac{\eta}{(N+1)\,\xi}\right)\right]\right\} \quad \text{wedge} \tag{109a}$$

$$= -\frac{5N^2+74N-3}{32}-\frac{(5N^2+10N+1)}{4(N+1)^2}\left(\frac{\eta}{\xi}\right)^2+\frac{5}{3(N+1)}\left(\frac{\eta}{\xi}\right)^3-\frac{11}{6(N+1)^2}\left(\frac{\eta}{\xi}\right)^4$$
$$-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left\{\frac{1}{4}+\frac{1}{\xi}-\frac{1}{\xi^2}-\frac{6}{\xi^4}+\frac{1}{(N+1)^2}\left(\frac{\eta}{\xi}\right)^2\right.$$
$$-\left[\frac{1}{2}-\frac{1}{\xi}-\frac{1}{\xi^2}-\frac{(\xi+2)\,\eta}{(N+1)\,\xi^2}\right]e^{-\xi}+\exp\left[-\xi\left(1-\left\{\frac{2\eta}{(N+1)\,\xi}\right\}^{\frac{1}{2}}\right)\right]\left[\frac{6}{\xi^4}-\frac{6}{\xi^3}\left(\frac{2\eta}{(N+1)\,\xi}\right)^{\frac{1}{2}}\right.$$
$$+\frac{6\eta}{(N+1)\,\xi^3}-\frac{2\eta}{(N+1)\,\xi^2}\left(\frac{2\eta}{(N+1)\,\xi}\right)^{\frac{1}{2}}+\frac{3(\xi+2)}{\xi^3}-\frac{3(\xi+2)}{\xi^2}\left(\frac{2\eta}{(N+1)\,\xi}\right)^{\frac{1}{2}}$$
$$\left.\left.+\frac{2\eta(\xi+2)}{(N+1)\,\xi^2}\right]\right\} \quad \text{cone.} \tag{109b}$$

These are complicated expressions; however, restricting interest to the surface pressure only, one obtains

$$p_3(\xi,0) = -(N^2+3N-1)-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)(1-e^{-\xi}+\xi e^{-\xi}) \quad \text{wedge} \tag{110a}$$

$$= -\frac{5N^2+74N-3}{32}-2\left(\frac{\lambda}{e^{\lambda/(N+1)}-1}-\frac{\lambda}{e^{\lambda/N}-1}\right)\left[\frac{1}{4}+\frac{1}{\xi}-\frac{1}{\xi^2}-\frac{6}{\xi^4}\right.$$
$$\left.-\left(\frac{1}{2}-\frac{1}{\xi}-\frac{4}{\xi^2}-\frac{6}{\xi^3}-\frac{6}{\xi^4}\right)e^{-\xi}\right] \quad \text{cone.} \tag{110b}$$

The non-equilibrium term again shows the change of value from zero at the leading edge to a finite value far downstream and thus makes the surface pressure relax from the frozen-flow value to the equilibrium-flow value. The relaxation, unlike that of density, is non-monotonic in both the wedge and cone cases. The surface pressure decreases to a minimum and then increases to reach the equilibrium-flow value far downstream. It is easy to show from (110) that the minimum point on the wedge is two relaxation lengths from the leading edge

($\xi = 2{\cdot}0$) and that on the cone it is about three relaxation lengths from the leading edge ($\xi = 2{\cdot}87$).

Figure 2 shows the relaxation of the surface pressure on a 20° cone in a free stream of oxygen at 300° K and a Mach number of 15. The upper curve is the result of the present analysis and the lower curve is the result of the numerical solution of the exact problem carried out by Sedney & Gerber. The two curves

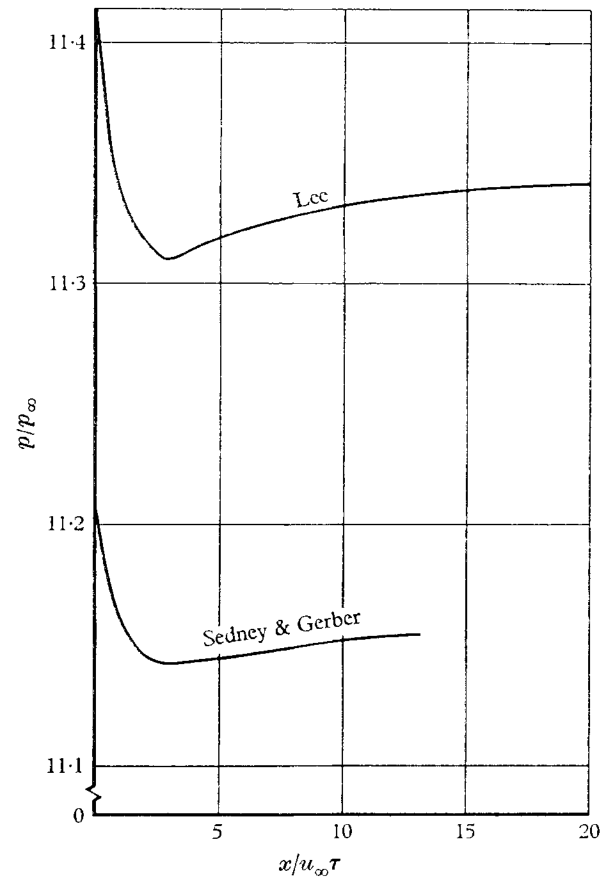

FIGURE 2. Surface-pressure distribution on a 20° cone in a free stream of oxygen at 300° K, $M_\infty = 15$.

agree, at least qualitatively. They agree especially well so far as the location of minimum pressure is concerned. Figure 3 compares the relaxation of the surface pressure on a 20° wedge with that on the 20° cone in the same free stream. It shows that the non-equilibrium effect on the wedge is more pronounced than on the cone, but otherwise the two curves are qualitatively similar.

6. Concluding remarks

The Newtonian or thin-shock-layer approximation is a powerful and frequently used tool in hypersonic-flow studies. In the present work it has been successfully employed to study the non-equilibrium effect due to molecular vibration. The results show that the classical Newtonian value for the various flow quantities is not affected to a first approximation. These classical values can be used in the rate equation to calculate the non-equilibrium variable. This is indeed practised in some engineering calculations. One must not generalize, however, when other rate processes are considered. The non-equilibrium effect enters as a relaxation term in the second approximation for the density, the lateral

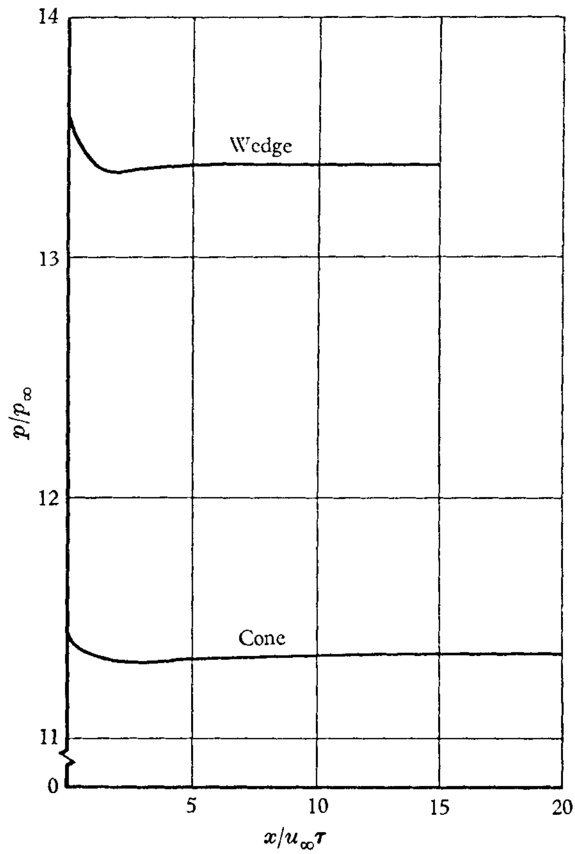

FIGURE 3. Surface-pressure distributions on a 20° wedge and a 20° cone in the same free stream.

velocity and the shock-wave shape, and in the third approximation for the pressure and the axial velocity. The relaxation effect is largely the same in the two-dimensional flow over a wedge and the axisymmetric flow over a cone.

The relaxation of the surface pressure is non-monotonic in both the flow over a wedge and that over a cone. This is in contrast to normal expectation of a monotonic relaxation derived from experience with uniformly relaxing cases. In the present case, the flow on different streamlines relaxes at different temperature, pressure and rate. It is possible that this difference in flow properties can be propagated to and thus have influence on the neighbouring streamlines before the final equilibrium state is reached. The influence should occur especially in cases similar to the present one in which the Mach number is high.

The entropy production, though not considered in the problem, can be calculated from the present results. It can be shown that even far downstream, where the flow reaches the equilibrium state, the entropy production has different values from streamline to streamline rather than a uniform equilibrium-flow value.

The author is indebted to Dr R. Sedney and Mr N. Gerber for furnishing the numerical solution. This work was supported by the Douglas Independent Research and Development Program.

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